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Imperfection Sensitivity of Conical Shells

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The sensitivity of isotropic conical shells to imperfection is considered, via the initial postbuckling analysis, on the basis of three different shell theories: Donnell's, Sanders's, and Timoshenko's. The conical shell was chosen as a representative case exhibiting the entire range of imperfection sensitivity. The procedure involves nonlinear partial differential equations, which are converted into a sequence of three linear sets. The latter are solved with the variables expanded in Fourier series in the circumferential direction and in finite differences in the axial directions. A general code is developed and used in studying the effect of higher exactness of the shell theory on the sensitivity behavior and in parametric analyses of the sensitivity of conical shells, especially with respect to the cone semivertex angle.

Nomenclature

A, B, D	=	membrane, coupling, and flexural rigidities
b	=	Koiter imperfection sensitivity parameters
G	=	geometric matrix
K	=	stiffness matrix
L, LL, LLL	=	linear, quadratic, and cubic differential operators
M_{xx}, $M_{\theta\theta}$, $M_{x\theta}$	=	bending moments
N_u, N_v, N_w	=	retained terms of truncated Fourier series
N_{xx}, $N_{\theta\theta}$, $N_{x\theta}$	=	membrane forces
$r(x)$	=	radius at the x coordinate (see Fig. 1)
Q_{ij}	=	transformed reduced stiffness
q_u, q_v, q_w	=	external distributed loading in the axial, circumferential, and normal directions
u, v, w	=	axial, circumferential, and normal displacements
x, θ	=	conical coordinates of the reference surface
z	=	outward normal coordinate
α	=	cone semivertex angle
ε^0_{xx}, $\varepsilon^0_{\theta\theta}$, $\gamma^0_{x\theta}$	=	strain of the reference surface
λ	=	buckling load parameter
ξ	=	perturbation parameter
χ_{xx}, $\chi_{\theta\theta}$, $\chi_{x\theta}$	=	change of curvatures
$(\,)_{,x}$, $(\,)_{,\theta}$	=	derivatives with respect to the axial and circumferential coordinate
(0), (1), (2)	=	prebuckling, buckling, and initial postbuckling states, respectively
$\dot{(\,)}$	=	derivatives with respect to the load parameter

Introduction

SHELL structures are widely used in aeronautical, marine, and civil engineering structures. The loss of stability by buckling of thin-shell structures is one of the most important and crucial failure phenomena. However, the behavior of shell-like structures under buckling is characterized by limit points rather than by bifurcation points (meaning that the theoretical buckling load is much higher than the real one), a fact due to their highly sensitivity to imperfection of the initial geometry. One of the main goals, in this field, is to find the various parameters that influence the shell's sensitivity, thereby improving the behavior of the structure as a whole.

In the present work, the conical shell was chosen as a representative case exhibiting the entire range of imperfection sensitivity: from the extremely sensitive cylindrical shell to the completely insensitive annular plate. By varying the cone vertex half-angle, conclusions can be drawn regarding the characteristic behavior and an insight gained into its mechanism.

Solution of shell structures by the linear theory has been widely investigated; see, for example, Love[1] and Donnell.[2] However, in many problems, such as large deformations and rotations, loss of stability, load-displacement relations, etc., the nonlinear effect is highly significant and must be accounted for, with the geometric kinematic theory modified accordingly. Most of the resulting approuches involve large displacements and moderate rotations [such as Timoshenko's (see Ref. 3), Flugge's,[4] and Sanders's[5]) and differ in the underlying kinematic assumptions. Comparison of different shell theories for buckling behavior,[6] for nonlinear behavior,[7,8] and for the imperfection sensitivity of laminated cylindrical shells shows pronounced discrepancies depending on the shell's aspect ratio. Hence, one of the objectives of this research was to investigate the validity of the kinematic assumptions used in three of the most common shell theories: Donnell's,[2] Sanders's,[5] and Timoshenko's.[3] Besides, their validity has not yet been investigated for conical shells.

During the past 50 years shells were designed under the criterion of the lower-bound design philosophy. This criterion recommends the empirical knockdown factor, chosen so that, when it is multiplied by the classical buckling load of the perfect shell, a lower bound for all available experimental data is obtained. For reliable results, and for fixing the level of these factors, insight into the postbuckling stage is essential in lightweight structural design.

With regard to structures characterized by limit points, there are two different main approaches in investigating their behavior:

1) Tracing of the entire nonlinear equilibrium path with emphasis on the level and direction of change of the stiffness during loading. This approach is extremely complicated and entails a heavy computational effort and, worst of all, it cannot cover all cases, as each new configuration has to be reanalyzed from the beginning.

2) Parametric study of the shell in terms of its sensitivity to imperfections and its rating according to this criterion. This approach was originally suggested by Koiter[9] and subsequently established by the following well-known research works: Budiansky and Hutchinson,[10] Budiansky,[11−13] Hutchinson and Amazigo,[14] Hutchinson and Frauenthal,[15] Cohen,[16] Fitch,[17] Fitch and Budiansky,[18] Arbocz and Hol,[19,20] and the review paper of Hutchinson and Koiter.[21] All of them used asymptotic formulation for the sensitivity parameters. Some of them applied it to specific engineering structures, mostly to cylindrical shells, for which only the simplest Donnell-type theory was used.

The main goal of this paper is to determine the parameters that affect the buckling load and the imperfection sensitivity. The conical shell is analyzed, as a representative of the characteristic behavior of shell structures, in terms of the Koiter sensitivity b-parameter.

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Very few research works have been done for the imperfection sensitivity of the conical shell; among them, the well-known are those of Zhang[22] and Zhang and Arbocz.[23] In those works laminated conical shells were investigated by the simplest Donnell's theory using w–F formulation, which could lead to inaccurate results (see Ref. 6). Furthermore, those works are limited to axisymmetric prebuckling, and the parametric study focuses on the influence of the laminated combinations on the imperfection sensitivity.

The nonlinear equilibrium differential equations for the three theories are derived in the basis of the kinematic approach, with the displacement components [axial (u), circumferential (v), and normal (w)] as the unknown dependent variables, and converted into three linear sets by the asymptotic technique. These sets are solved through expansion of the dependent variables in Fourier series in the circumferential direction and in finite differences in the axial direction. The Galerkin procedure is then applied for minimizing the error due to the truncated form of the series. (For the sake of completeness, some parts of the procedure are briefly repeated.) A general computer code was developed and used for a wide range of parametric study of the buckling and sensitivity behavior.

It was found that the major changes in the sensitivity behavior are concentrated at the edges of the semivertex angle scale.

Governing Equations

Kinematics

Let (x, θ) be the conical coordinates of the reference surface (Fig. 1) and z the outward normal coordinate. Recourse to the Kirchhoff–Love hypothesis leaves only three dependent variables, namely the displacement u, v, and w in the x, θ, and z directions, respectively. Under the three kinematic approaches in question, the strain displacement can be written as

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{\chi\} \tag{1}$$

where $\{\varepsilon^0\}$ and $\{\chi\}$ are, respectively, the strain of the reference surface and change-of-curvature vectors, composed as follows:

$$\varepsilon^0_{xx} = u_{,x} + \frac{1}{2}w^2_{,x} + \delta_2\frac{v^2_{,x}}{2}$$

$$\varepsilon^0_{\theta\theta} = \frac{v_{,\theta}}{r(x)} + \frac{w}{r(x)}\cos(\alpha) + \frac{u}{r(x)}\sin(\alpha) + \frac{w^2_{,\theta}}{2r(x)^2}$$

$$+ \delta_1\frac{\cos(\alpha)}{r(x)^2}\left[\frac{1}{2}v^2\cos(\alpha) - vw_{,\theta}\right] + \frac{\delta_2}{2}\left[\frac{v_{,\theta} + w\cos(\alpha)}{r(x)}\right]^2$$

$$\gamma^0_{x\theta} = \frac{u_{,\theta}}{r(x)} + v_{,x} + \frac{w_{,x}w_{,\theta}}{r(x)} - \frac{v}{r(x)}\sin(\alpha)$$

$$- \delta_1\frac{\cos(\alpha)}{r(x)}vw_{,x} + \delta_2\frac{v_{,x}v_{,\theta}}{r(x)} \tag{2}$$

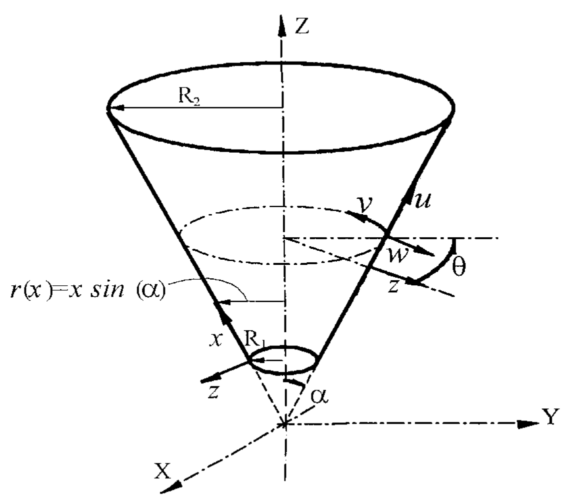

Fig. 1 Geometry and sign convention for coordinates and displacements.

$$\chi_{xx} = -w_{,xx}$$

$$\chi_{\theta\theta} = -\frac{w_{,\theta\theta}}{r(x)^2} - \frac{w_{,x}}{r(x)}\sin(\alpha) + \delta_1\frac{\cos(\alpha)}{r(x)^2}v_{,\theta}$$

$$\chi_{x\theta} = -\frac{w_{,x\theta}}{r(x)} + \frac{w_{,\theta}}{r(x)^2}\sin(\alpha) + \delta_1\frac{\cos(\alpha)}{r(x)}\left[\frac{1}{2}v_{,x} - \frac{v}{r(x)}\sin(\alpha)\right] \tag{3}$$

where δ_1 and δ_2 are introduced for representing the specific shell theories:

$\delta_1 = \delta_2 = 0$		for	Donnell's
$\delta_1 = 1,$	$\delta_2 = 0$	for	Sanders's
$\delta_1 = \delta_2 = 1$		for	Timoshenko's

The Timoshenko approach contains some additional terms, but the most dominant one is due to $\delta_2 = 1$, especially under axial compression. Furthermore, the terms of δ_2 in $\gamma^0_{x\theta}$ are sometimes neglected, depending on the external loading type, and those in $\varepsilon^0_{\theta\theta}$ are always neglected.

By letting $\alpha = 0$ and $r(x) = r$ the kinematic relations degenerate to cylinder equations,[7] and by letting $\alpha = 90$ deg and $r(x) = x$, to an annular plate.[24]

Constitutive Equations

Under the classical laminate theory, the constitutive equation reads

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{Bmatrix} \varepsilon^0 \\ \chi \end{Bmatrix} \tag{4}$$

$\{\boldsymbol{N}\} = \{N_{xx}, N_{\theta\theta}, N_{x\theta}\}$ and $\{\boldsymbol{M}\} = \{M_{xx}, M_{\theta\theta}, M_{x\theta}\}$ being the membrane force and bending moment vectors. The coefficients of the elastic matrix are given by

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{n}\int_{h_{k-1}}^{h_k} Q_{ij}\cdot(1, z, z^2)\,\mathrm{d}z \tag{5}$$

A_{ij}, B_{ij}, and D_{ij} being, respectively, the membrane, coupling, and flexural rigidities, and Q_{ij} the laminate transformed reduced stiffness.

Equilibrium Equations

The equilibrium equations and the appropriate boundary conditions are derived by applying the variational principle:

$$\delta\pi = \oint_\theta\int_x \left[N_{xx}\delta\varepsilon^0_{xx} + N_{\theta\theta}\delta\varepsilon^0_{\theta\theta} + N_{x\theta}\delta\gamma^0_{x\theta} + M_{xx}\delta\chi_{xx} + M_{\theta\theta}\delta\chi_{\theta\theta}\right.$$

$$\left. + 2M_{x\theta}\delta\chi_{x\theta}\right]\mathrm{d}x\,\mathrm{d}\theta - \oint_\theta\int_x (q_u\delta u + q_v\delta v + q_w\delta w)\,\mathrm{d}x\,\mathrm{d}\theta$$

$$- \oint_\theta(\bar{N}_{xx}\delta u + \bar{N}_{x\theta}\delta v + \bar{Q}\delta w + \bar{M}_{xx}\delta w_{,x})\,\mathrm{d}\theta\Big|_{x=0}^{x=l} \tag{6}$$

where $\bar{N}_{xx}$, $\bar{N}_{x\theta}$, $\bar{Q}$, and $\bar{M}_{xx}$ are, respectively, the axial, torsional, shearing forces, and the bending moment applied through the boundaries.

With Eqs. (2) and (3) substituted into Eq. (6), application of the divergence theorem yields the following equilibrium equations:

$$N_{xx,x} + \frac{N_{x\theta,\theta}}{r(x)} + \frac{\sin(\alpha)}{r(x)}[N_{xx} - N_{\theta\theta}] + q_u = 0$$

$$N_{x\theta,x} + \frac{N_{\theta\theta,\theta}}{r(x)} + \frac{2\sin(\alpha)}{r(x)}N_{x\theta} + \delta_1\frac{\cos(\alpha)}{r(x)}\left\{\frac{N_{\theta\theta}}{r(x)}[w_{,\theta} - v\cos(\alpha)]\right.$$

$$-\delta_3\frac{N_{\theta\theta}}{r(x)}[w_{,\theta} - v\cos(\alpha)] + N_{x\theta}w_{,x} + M_{x\theta,x} + \frac{M_{\theta\theta,\theta}}{r(x)}$$

$$\left. + \frac{2\sin(\alpha)}{r(x)}M_{x\theta}\right\} + \delta_2\left[(N_{xx}v_{,x})_{,x} + \frac{N_{xx}v_{,x}}{r(x)}\sin(\alpha)\right.$$

$$+\frac{(N_{x\theta}v_{,x})_{,\theta}}{r(x)}+\left(\frac{N_{x\theta}v_{,\theta}}{r(x)}\right)_{,x}+\frac{N_{x\theta}v_{,\theta}}{r(x)^2}\sin(\alpha)\Bigg]+q_v=0$$

$$M_{xx,xx}+\frac{M_{\theta\theta,\theta\theta}}{r(x)^2}+\frac{2M_{x\theta,x\theta}}{r(x)}-\frac{N_{\theta\theta}}{r(x)}\cos(\alpha)$$

$$+\frac{\sin(\alpha)}{r(x)}\left[2M_{xx,x}-M_{\theta\theta,x}+\frac{2M_{x\theta,\theta}}{r(x)}\right]+\frac{1}{r(x)}[r(x)N_{xx}w_{,x}$$

$$+N_{x\theta}w_{,\theta}]_{,x}+\frac{1}{r(x)}\left[N_{x\theta}w_{,x}+\frac{N_{\theta\theta}w_{,\theta}}{r(x)}\right]_{,\theta}$$

$$-\delta_1\frac{\cos(\alpha)}{r(x)}\left[\frac{1}{r(x)}(N_{\theta\theta}v)_{,\theta}+(N_{x\theta}v)_{,x}\right]+q_w=0 \qquad (7)$$

with the following boundary conditions:

$$N_{xx}=\bar{N}_{xx} \qquad \text{or} \qquad u=\bar{u}$$

$$N_{x\theta}+\delta_1\frac{\cos(\alpha)}{r(x)}M_{x\theta}$$

$$+\delta_2\left[N_{xx}v_{,x}+\frac{N_{x\theta}v_{,\theta}}{r(x)}\right]=\bar{N}_{x\theta} \qquad \text{or} \qquad v=\bar{v}$$

$$M_{xx,x}+\frac{2M_{x\theta,\theta}}{r(x)}+N_{xx}w_{,x}+\frac{N_{x\theta}w_{,\theta}}{r(x)}+\frac{\sin(\alpha)}{r(x)}(M_{xx}-M_{\theta\theta})$$

$$-\delta_1\frac{\cos(\alpha)}{r(x)}N_{x\theta}v=\bar{Q} \qquad \text{or} \qquad w=\bar{w}$$

$$M_{xx}=\bar{M}_{xx} \qquad \text{or} \qquad w_{,x}=\bar{w}_{,x} \qquad (8)$$

where $\bar{()}$ denotes prescribed displacements (or forces) at the boundary. Note that the Timoshenko approach omits some terms of δ_2 from Eqs. (7b) and (8b).

In Eq. (7), δ_3 is a corrector or factor introduced for the hydrostatic load case. The load direction of curved shells under hydrostatic pressure changed during buckling.[25] There are two possibilities concerning the behavior of the loads. In case 1 (the real load case), it is assumed that the load remains normal to the deflected reference axis, here $\delta_1=\delta_3=1$. In case 2, it is assumed that the load remains parallel to its original direction; here $\delta_1=0$. For a thin ring under hydrostatic pressure, case 1 yields a buckling load lower than that of case 2 by 33%. Furthermore, case 1 represents Sanders's theory and case 2 Donnell's. Note that the correction for the first loading case could be also achieved directly from the potential energy.[26]

Using Eqs. (2–4) and (7), the equilibrium equations in terms of the displacement function read

$$L_h^{[1]}(w)+L_q^{[1]}(v)+L_e^{[1]}(u)+LL_g^{[1]}(w,w)$$

$$+\delta_1\left\lfloor LL_m^{[1]}(v,v)+LL_n^{[1]}(w,v)\right\rfloor+q_u=0$$

$$L_h^{[2]}(w)+L_q^{[2]}(v)+L_e^{[2]}(u)+LL_g^{[2]}(w,w)+\delta_1\big[LL_m^{[2]}(v,v)$$

$$+LL_n^{[2]}(w,v)+LL_s^{[2]}(w,u)+LL_t^{[2]}(u,v)$$

$$+LLL_p^{[2]}(w,v,w)+LLL_f^{[2]}(v,w,v)+LLL_l^{[2]}(v,v,v)$$

$$+LLL_y^{[2]}(w,w,w)\big]+q_v=0$$

$$L_h^{[3]}(w)+L_q^{[3]}(v)+L_e^{[3]}(u)+LL_g^{[3]}(w,w)+LL_n^{[3]}(w,v)$$

$$+LL_s^{[3]}(w,u)+LLL_y^{[3]}(w,w,w)+\delta_1\big[LL_m^{[3]}(v,v)$$

$$+LL_t^{[3]}(u,v)+LLL_p^{[3]}(w,v,w)+LLL_f^{[3]}(v,w,v)$$

$$+LLL_l^{[3]}(v,v,v)\big]+q_w=0 \qquad (9)$$

$L^{[e]}$, $LL^{[e]}$, and $LLL^{[e]}$, $[e]=1,2,3$, are, respectively, linear, quadratic, and cubic differential operators. They are given by[6]

$$L_p^{[e]}(S)=\sum_{i=0}^{4}\sum_{j=0}^{4-i}p_{ij}^{[e]}\frac{\partial^{(i+j)}S}{\partial x^{(i)}\partial\theta^{(j)}}, \qquad p=h,q,e$$

$$LL_p^{[e]}(S,T)=\sum_{i=0}^{3}\sum_{j=0}^{3-i}\sum_{\ell=0}^{3}\sum_{k=0}^{3-\ell}p_{ijk\ell}^{[e]}\frac{\partial^{(i+j)}S}{\partial x^{(i)}\partial\theta^{(j)}}\frac{\partial^{(\ell+k)}T}{\partial x^{(\ell)}\partial\theta^{(k)}}$$

$$p=g,m,n,s,t$$

$$LLL_p^{[e]}(S,T,S)$$

$$=\sum_{i=0}^{2}\sum_{j=0}^{2-i}\sum_{\ell=0}^{2}\sum_{k=0}^{2-\ell}\sum_{m=0}^{1}\sum_{n=0}^{1-m}p_{ijk\ell mn}^{[e]}\frac{\partial^{(i+j)}S}{\partial x^{(i)}\partial\theta^{(j)}}\frac{\partial^{(\ell+k)}T}{\partial x^{(\ell)}\partial\theta^{(k)}}\frac{\partial^{(m+n)}S}{\partial x^{(m)}\partial\theta^{(n)}}$$

$$p=y,l,p,f \qquad (10)$$

where $p_{ij}^{[e]}$, $p_{ijkl}^{[e]}$, and $p_{ijklmn}^{[e]}$ are, respectively, the coefficients of the elastic parameters, functions of A_{ij}, B_{ij}, D_{ij}, of the radius $r(x)$, and of δ_1, δ_2, and δ_3; S and T are symbolic variables assigned to u, v, and w.

The boundary conditions in terms of the displacement functions are

$$L_{b\cdot e}^{[1]}(u)+L_{b\cdot q}^{[1]}(v)+L_{b\cdot h}^{[1]}(w)+LL_{b\cdot g}^{[1]}(w,w)$$

$$+\delta_1\big[LL_{b\cdot m}^{[1]}(v,v)+LL_{b\cdot n}^{[1]}(w,v)\big]=\bar{N}_{xx} \qquad \text{or} \qquad u=\bar{u}$$

$$L_{b\cdot e}^{[2]}(u)+L_{b\cdot q}^{[2]}(v)+L_{b\cdot h}^{[2]}(w)+L_{b\cdot g}^{[2]}(w,w)+\delta_1\big[LL_{b\cdot m}^{[2]}(v,v)$$

$$+LL_{b\cdot n}^{[2]}(w,v)\big]+\delta_2\big[LL_{b\cdot m}^{[2]}(v,v)+LL_{b\cdot t}^{[2]}(u,v)$$

$$+LL_{b\cdot n}^{[2]}(w,v)+LLL_{b\cdot l}^{[2]}(v,v,v)+LLL_{b\cdot f}^{[2]}(v,w,v)$$

$$+LLL_{b\cdot p}^{[2]}(w,v,w)\big]=\bar{N}_{x\theta} \qquad \text{or} \qquad v=\bar{v}$$

$$L_{b\cdot e}^{[3]}(u)+L_{b\cdot q}^{[3]}(v)+L_{b\cdot h}^{[3]}(w)+LL_{b\cdot g}^{[3]}(w,w)+LL_{b\cdot s}^{[3]}(w,u)$$

$$+LL_{b\cdot n}^{[3]}(w,v)+LLL_{b\cdot y}^{[3]}(w,w,w)+\delta_1\big[LL_{b\cdot m}^{[3]}(v,v)$$

$$+LL_{b\cdot t}^{[3]}(u,v)+LLL_{b\cdot p}^{[3]}(w,v,w)$$

$$+LLL_{b\cdot f}^{[3]}(v,w,v)\big]=\bar{Q} \quad \text{or} \quad w=\bar{w}$$

$$L_{b\cdot e}^{[4]}(u)+L_{b\cdot q}^{[4]}(v)+L_{b\cdot h}^{[4]}(w)+LL_{b\cdot g}^{[4]}(w,w)+\delta_1\big[LL_{b\cdot m}^{[4]}(v,v)$$

$$+LL_{b\cdot n}^{[4]}(w,v)\big]=\bar{M}_{xx} \qquad \text{or} \qquad w_{,x}=\bar{w}_{,x}$$

$$(11)$$

Buckling Equations

The buckling equations are straightforward and derived with the aid of the perturbation technique:

$$\begin{Bmatrix}u\\v\\w\end{Bmatrix}=\begin{Bmatrix}u^{(0)}\\v^{(0)}\\w^{(0)}\end{Bmatrix}+\xi\begin{Bmatrix}u^{(1)}\\v^{(1)}\\w^{(1)}\end{Bmatrix} \qquad (12)$$

Applying Eqs. (12), perturbation of the differential operators yields

$$L(S)=L\big(S^{(0)}\big)+\xi L\big(S^{(1)}\big)$$

$$LL(S,T)=LL\big(S^{(0)},T^{(0)}\big)+\xi\big\lfloor LL\big(S^{(0)},T^{(1)}\big)$$

$$+LL\big(S^{(1)},T^{(0)}\big)\big\rfloor+\xi^2[\cdots]$$

$$LLL(S,T,S)=LLL\big(S^{(0)},T^{(0)},S^{(0)}\big)+\xi\big[LLL\big(S^{(0)},T^{(0)},S^{(1)}\big)$$

$$+LLL\big(S^{(0)},T^{(1)},S^{(0)}\big)+LLL\big(S^{(1)},T^{(0)},S^{(0)}\big)\big]+\xi^2[\cdots]$$

$$(13)$$

Substitution of Eqs. (12) and (13) into Eq. (9) yields the partial differential equations of the prebuckling state

$$L^{[e]}\left(S^{(0)}\right)+LL^{[e]}\left(S^{(0)},T^{(0)}\right)+LLL^{[e]}\left(S^{(0)},T^{(0)},S^{(0)}\right)=P^{[e]}\qquad e=1,2,3 \tag{14}$$

and the buckling state

$$L^{[e]}\left(S^{(1)}\right)+\lambda\left[LL^{[e]}\left(S^{(0)},T^{(1)}\right)+LL^{[e]}\left(S^{(1)},T^{(0)}\right)\right]+\lambda^2\left[LLL^{[e]}\left(S^{(1)},T^{(0)},S^{(0)}\right)+LLL^{[e]}\left(S^{(0)},T^{(1)},S^{(0)}\right)+LLL^{[e]}\left(S^{(0)},T^{(0)},S^{(1)}\right)\right]=0,\qquad e=1,2,3 \tag{15}$$

The sum of the indices of the prebuckling state is zero, whereas that in each operator of the buckling is unity.

The set of partial differential equations is first reduced to one of ordinary ones by separation of the variables, namely,

$$u(x,\theta)=\sum_{m=0}^{2Nu}u_m(x)g_m(\theta),\qquad v(x,\theta)=\sum_{m=0}^{2Nv}v_m(x)g_m(\theta)$$

$$w(x,\theta)=\sum_{m=0}^{2Nw}w_m(x)g_m(\theta) \tag{16}$$

where $2Nu$, $2Nv$, and $2Nw$ are the numbers of retained terms in the relevant truncated series and

$$g_m(\theta)=\begin{cases}\cos \mathrm{im}\,\theta, & m=0,1,2,3,\ldots,N_3\\ \sin \mathrm{im}\,\theta, & m=N_3+1,\ldots,2N_3\end{cases} \tag{17}$$

i denoting the characteristic circumferential wave number.[27] $N_3=N_u$, N_v, or N_w according to the equation number. On application of the Galerkin procedure for minimizing the errors due to the truncated form of the series, the following integrals must vanish:

$$\int_0^{2\pi}[\text{equilibrium in } u,\text{ Eq. (7a)}]g_p(\theta)\,\mathrm{d}\theta=0\qquad p=0,1,2,\ldots,2N_u$$

$$\int_0^{2\pi}[\text{equilibrium in } v,\text{ Eq. (7b)}]g_p(\theta)\,\mathrm{d}\theta=0\qquad p=0,1,2,\ldots,2N_v$$

$$\int_0^{2\pi}[\text{equilibrium in } w,\text{ Eq. (7c)}]g_p(\theta)\,\mathrm{d}\theta=0\qquad p=0,1,2,\ldots,2N_w \tag{18}$$

$g_p(\theta)$ are the weighting functions, chosen as $\cos(\mathrm{im}\,\theta)$ and $\sin(\mathrm{im}\,\theta)$. The Galerkin procedure is applied in the same way for the boundary condition.

The present theory assumes a linear prebuckling behavior, which yields an eigenproblem.

Finally, a central finite difference scheme is used to reduce the ordinary differential equations to the following algebraic one:

For the prebuckling state,

$$[K]\{\mathbf{Z}\}=\{P\} \tag{19}$$

and for the buckling state,

$$\{[K]+\lambda[G]\}\{\mathbf{Z}\}=0 \tag{20}$$

where $\mathbf{Z}$ is an unknown vector consisting of u, v, w, $u_{,xx}$, $v_{,xx}$, and $w_{,xx}$. Equation (20) is an eigenvalue problem in which λ represents the buckling load parameters and $\mathbf{Z}$ the buckling mode.

Initial Postbuckling Equation

The imperfection sensitivity parameters are derived using the initial postbuckling equations via asymptotic expansion. This determines whether the load initially increases or decreases after buckling. Accordingly, the displacement, strain and stress vectors are expanded as follows:

$$\begin{Bmatrix}u\\v\\w\end{Bmatrix}=\lambda\begin{Bmatrix}u^{(0)}\\v^{(0)}\\w^{(0)}\end{Bmatrix}+\xi\begin{Bmatrix}u^{(1)}\\v^{(1)}\\w^{(1)}\end{Bmatrix}+\xi^2\begin{Bmatrix}u^{(2)}\\v^{(2)}\\w^{(2)}\end{Bmatrix}+\cdots \tag{21}$$

λ being a load parameter deviating from the classical buckling load λ_c. Substitution of Eq. (21) into Eqs. (9) yields distinct sets of equations for the three states, namely the zero order, first order, and higher order, respectively.

The first two states have already been treated, and so here only the postbuckling state is considered; it is formulated as

$$L^{[e]}\left(S^{(2)}\right)+\lambda_c\left[LL^{[e]}\left(S^{(0)},T^{(2)}\right)+LL^{[e]}\left(S^{(2)},T^{(0)}\right)\right]+\lambda_c^2\left[LLL^{[e]}\left(S^{(2)},T^{(0)},S^{(0)}\right)+LLL^{[e]}\left(S^{(0)},T^{(2)},S^{(0)}\right)+LLL^{[e]}\left(S^{(0)},T^{(0)},S^{(2)}\right)\right]=\mathrm{RHS}^{[e]},\qquad e=1,2,3 \tag{22}$$

where $L^{[e]}$, $LL^{[e]}$, $LLL^{[e]}$, S, and T are again as per Eqs. (9) and (10); for example,

$$L^{[1]}\left(S^{(2)}\right)=L_e^{[1]}\left(u^{(2)}\right)+L_q^{[1]}\left(v^{(2)}\right)+L_h^{[1]}\left(w^{(2)}\right)$$

$$LL^{[1]}\left(S^{(0)},T^{(2)}\right)+LL^{[1]}\left(S^{(2)},T^{(0)}\right)=2L_g^{[1]}\left(w^{(0)},w^{(2)}\right)+\delta_1\left[2L_m^{[1]}\left(v^{(0)},v^{(2)}\right)+L_n^{[1]}\left(w^{(0)},v^{(2)}\right)+L_n^{[1]}\left(w^{(2)},v^{(0)}\right)\right]$$

$$LLL^{[1]}=0 \tag{23}$$

$\mathrm{RHS}^{[e]}$, the right-hand-side differential operators of equation e, is given in symbolic form as

$$\mathrm{RHS}^{[e]}=LL^{[e]}\left(S^{(1)},T^{(1)}\right)+\lambda_c\left[LLL^{[e]}\left(S^{(1)},T^{(1)},S^{(0)}\right)+LLL^{[e]}\left(S^{(0)},T^{(1)},S^{(1)}\right)+LLL^{[e]}\left(S^{(1)},T^{(0)},S^{(1)}\right)\right]\qquad e=1,2,3 \tag{24}$$

Here the sum of the indices of the operators is two. $S^{(0)}$, $T^{(0)}$ and $S^{(1)}$, $T^{(1)}$ are, respectively, the symbolic known variables of the prebuckling and buckling states, for example:

$$\mathrm{RHS}^{[1]}=-L_g^{[1]}\left(w^{(1)},w^{(1)}\right)-\delta_1\left\lfloor L_m^{[1]}\left(v^{(1)},v^{(1)}\right)+L_n^{[1]}\left(w^{(1)},v^{(1)}\right)\right\rfloor \tag{25}$$

The postbuckling-state (second order) $w^{(2)}$, $v^{(2)}$, and $u^{(2)}$ are obtained from the linear equation (22). Because $\mathrm{RHS}^{[e]}$ represents the quadratic level of the known buckling mode, the postbuckling state obviously depends on c^2, c being the scalar by which the buckling mode is normalized.

In the case of nonlinear behavior the expansion of the prebuckling state near the bifurcation point (λ_c) reads

$$\begin{Bmatrix}u^{(0)}\\v^{(0)}\\w^{(0)}\end{Bmatrix}=\begin{Bmatrix}u_c^{(0)}\\v_c^{(0)}\\w_c^{(0)}\end{Bmatrix}+(\lambda-\lambda_c)\begin{Bmatrix}\dot{u}_c^{(0)}\\\dot{v}_c^{(0)}\\\dot{w}_c^{(0)}\end{Bmatrix}+\cdots \tag{26}$$

where

$$\dot{()}=\frac{\partial()}{\partial\lambda}$$

Using Eqs. (21) and applying the variational principle following Cohen,[16] the load parameter is obtained as

$$\lambda/\lambda_c=1+b\xi^2+\cdots \tag{27}$$

The coefficient of first order vanishes due to the periodicity of the buckling mode in the circumferential direction; b is the well-known

Koiter imperfection-sensitivity parameter (a positive value indicating that the shell is insensitive, a negative value measuring the level of sensitivity) given by

$$\mathrm{b} = -\frac{\sigma_2 \cdot L_2(u_1) + 2\sigma_1 \cdot L_{11}(u_1, u_2)}{\lambda_c[2\sigma_1 \cdot L_{11}(\dot{u}_c, u_1) + \dot{\sigma}_c \cdot L_2(u_1)]} \tag{28}$$

This is the general formula for the dead load case obtained by Cohen[16] and Fitch[17] and later by Arbocz and Hol.[19]

For the case of a linear prebuckling state Budiansky and Hutchinson[10] and Hutchinson and Budiansky[28] derived the well-known formula

$$\mathrm{b} = -\frac{\sigma_2 \cdot L_2(u_1) + 2\sigma_1 \cdot L_{11}(u_1, u_2)}{\lambda_c \sigma_0 \cdot L_2(u_1)} \tag{29}$$

The main discrepancy between these two formulas is the term $2\sigma_1 \cdot L_{11}(u_0, u_1)$ in the denominator, omitted by Budiansky and Hutchinson. In most cases it is indeed negligible. Exceptions are the case of a spherical cap under concentrated load[17] and our own case of a conical shell under hydrostatic pressure (see next section). In both cases, due to the dominant effect of $2\sigma_1 \cdot L_{11}(u_0, u_1)$, the denominator vanishes and the sensitivity parameter, b, becomes singular.

For a conical shell, with the variables u, v, w, the operator will be

$$\begin{aligned}\sigma_i \cdot L_{11}(u_j, u_k) = \int_a^b \int_0^{2\pi} \Bigg\{ & N_{xx}^{(i)}\left[w_{,x}^{(j)} w_{,x}^{(k)} + \delta_2 v_{,x}^{(j)} v_{,x}^{(k)}\right] \\ & + N_{\theta\theta}^{(i)}\left[\frac{w_{,\theta}^{(j)} w_{,\theta}^{(k)}}{r(x)^2} + \frac{\delta_1 \cos(\alpha)}{r(x)^2}\left(\cos(\alpha) v^{(j)} v^{(k)} - v^{(j)} w_{,\theta}^{(k)} - v^{(k)} w_{,\theta}^{(j)}\right)\right] + 2N_{x\theta}^{(i)}\left[\frac{w_{,x}^{(j)} w_{,\theta}^{(k)}}{2r(x)} \right. \\ & \left. + \frac{w_{,x}^{(k)} w_{,\theta}^{(j)}}{2r(x)} - \frac{\delta_1 \cos(\alpha)}{2r(x)}\left(v^{(j)} w_{,x}^{(k)} + v^{(k)} w_{,x}^{(j)}\right) + \frac{\delta_2}{2r(x)}\left(v_{,x}^{(j)} v_{,\theta}^{(k)} + v_{,x}^{(k)} v_{,\theta}^{(j)}\right)\right]\Bigg\} \,\mathrm{d}\theta\, \mathrm{d}x, \qquad i, j, k = 0, 1, 2\end{aligned} \tag{30}$$

Here the superscripts (i), (j), and (k) denote the appropriate state [(0), prebuckling; (1), buckling; and (2), initial postbuckling].

Results and Discussion

For the procedure outlined, a general-purpose computer code ISOCS (Imperfection Sensitivity of Conical Shells) was written, covering the buckling and initial postbuckling behavior of any isotropic conical shell under arbitrary loading. The code, which incorporates the three comparative shell theories, is specially designed for examining the accuracy of the imperfection sensitivity parameter, over a wide range of shell aspect ratios.

The main object of the study is identification and analysis of the accuracy of the parameters with respect to the cone vertex half-angle α (hereinafter referred to as the "angle"). For this purpose an isotropic conical shell was taken, with data as follows: modulus of elasticity $E = 1.404 \times 10^{11}$ N/m^2, Poisson's ratio $\nu = 0.2$, shorter radius of the truncated cone $R_1 = 1.27$ m, and thickness $t = 0.0127$ m ($R_1/t = 100$). The study covered torsion, axial compression, and hydrostatic pressure.

Torsion

The torsional load was applied through the boundary conditions by setting $N_{x\theta} = \bar{N}_{x\theta}$ at the narrower end. First, the accuracy of the different shell theories was examined. The buckling modes according to the theories are plotted in Fig. 2 for a clamped shell ($w = w_{,x} = u = v = 0$), $l/R_1 = 30$. It is seen that, even when there is no discrepancy between the buckling loads obtained by the various theories, the buckling modes are quite different. The buckling loads (see the assigned critical values in Fig. 2) obtained through Donnell's and Sanders's (or Timoshenko's) theories differ only up to $\alpha = 20$ deg over the whole range of l/R_1, the significant discrepancy at $\alpha = 0$ deg (cylinder). As can be expected, the more accurate the theory (Sanders, Timoshenko), the lower the buckling load. The discrepancy between the theories increases with the l/R_1 ratio,[29] whereas the effect on the sensitivity b-parameter is the opposite.

In Fig. 3 the torsional buckling load and the sensitivity b-parameter are plotted vs the angle (α), for the clamped boundary condition and for different l/R_1 ratios. It is seen that in all cases the maximum buckling load is obtained around $\alpha \cong 40$ deg. At the same angle the buckling wave number is maximal,[29] and one can say that for a conical shell under torsion this is the optimal angle. Furthermore, the lower the l/R_1 ratio, the higher the buckling load and the more sensitive to imperfection. Transition from sensitive to insensitive behavior appears at small angles as the l/R_1 ratio decreases. Moreover, as the l/R_1 ratio increases the effect of the angle on the sensitivity b-parameter decreases, as can be seen in Fig. 4. The sensitivity behavior is characterized by transition of the wave

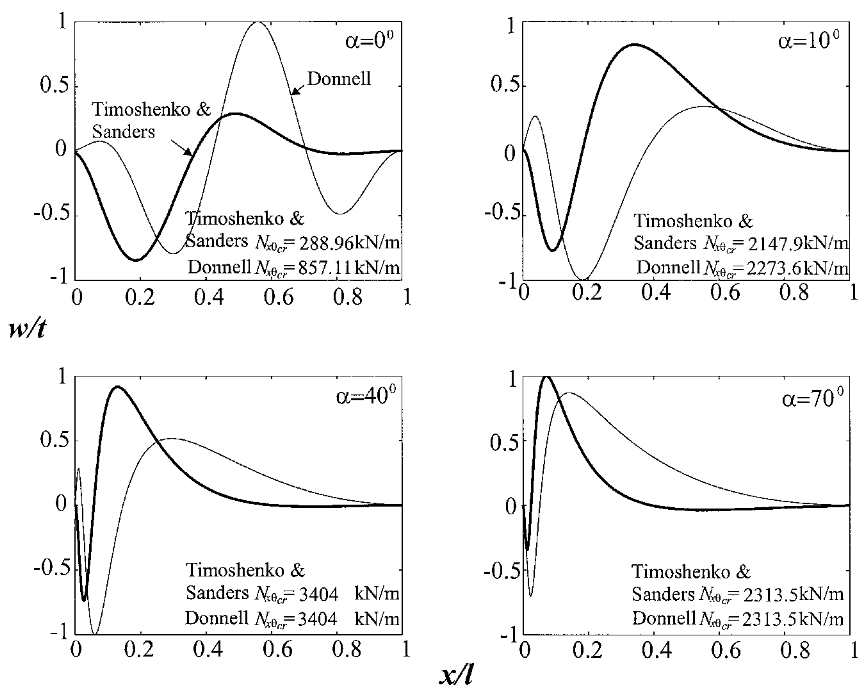

Fig. 2 Torsional buckling mode of transverse w-displacement for $l/R_1 = 30$.

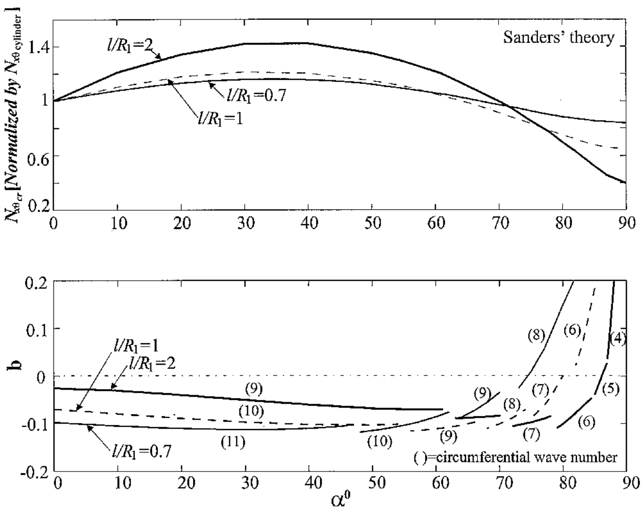

Fig. 3 Torsional buckling load and sensitivity b-parameter vs cone vertex half-angle for clamped conical shell according to Sanders's shell theory.

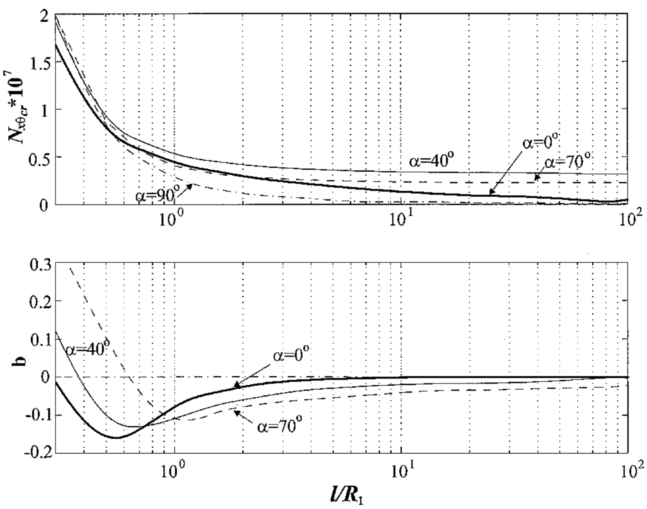

Fig. 4 Torsional buckling load and sensitivity b-parameter vs l/R_1 ratio.

number (depending on the angle) as shown in Fig. 3 by the numbers in the parentheses. Generally, this change yields a discontinuity in the sensitivity b-parameter. This phenomenon is familiar in the case of spherical caps under axisymmetric load[17,18] and was observed here for the first time for a conical shell.

Axial Compression

The accuracy of the shell theories for the axial compression case (applied by setting $N_{xx} = \bar{N}_{xx}$ at the narrower edge) is shown in Fig. 5. The axial buckling load (normalized by the buckling load of a cylindrical shell according to Donnell's theory, $N_{xx\,\text{cr}} = 10{,}454$, 10,492, 10,498, 10,511 kN/m for $l/R_1 = 1, 5, 10, 100$, respectively) is plotted vs the angle for different l/R_1 ratios. A significant discrepancy between the theories was obtained only for a long cylinder ($l/R_1 = 100$, $\alpha = 0$ deg), Sanders's and Timoshenko's theories yielding much lower buckling loads than Donnell's theory. Already at $\alpha = 1$ deg the buckling loads by the different theories are almost the same. Furthermore, the buckling mode is also affected by the shell theory.[29] It is well known that Donnell's theory is inaccurate for high l/R ratios, but on the slightest deviation from cylindrical shell the buckling loads obtained by the three theories are almost equal, so that Donnell's theory also yields an accurate result.

The sensitivity parameter according to three theories is plotted in Fig. 6 for a simply supported conical shell, $l/R_1 = 100$. It is seen that the discrepancy between the different theories also affects the sensitivity b-parameter, but again even at $\alpha = 1$ deg the difference is

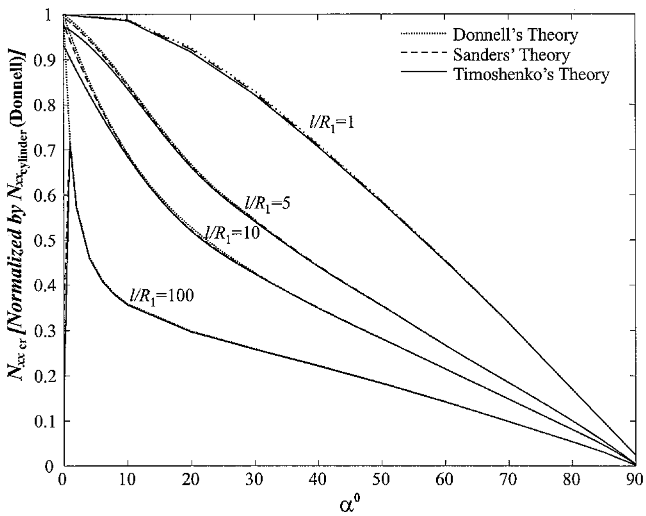

Fig. 5 Axial buckling load vs cone vertex half-angle for different l/R_1 ratios according to three shell theories.

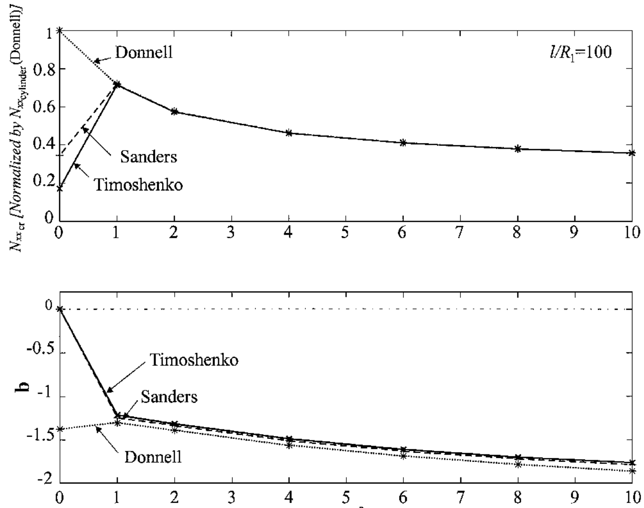

Fig. 6 Axial buckling load and sensitivity b-parameter vs cone vertex half-angle for simply supported ($\nu = 0$) conical shell.

negligible. Sanders's and Timoshenko's theories yield an insensitive cylinder, whereas Donnell's theory yields a highly sensitive shell. The most accurate theory yields the lowest sensitivity b-parameter.

The effect of the in-plane boundary conditions on the buckling load and on the sensitivity b-parameter is plotted in Figs. 7 and 8 for a simply supported and a clamped conical shell, respectively. In the first case this effect is quite pronounced in both respects, and the condition $N_{x\theta} = 0$ yields lower imperfection sensitivity. In the second case, however, only the sensitivity b-parameter is affected. In both cases the influence of the inplane boundary conditions is limited up to $\alpha = 20$ deg, and one can say that this angle represents the transition from sensitive characteristic behavior to total insensitivity.

Hydrostatic Pressure

In Fig. 9 the buckling load and the sensitivity b-parameter are plotted vs the l/R_1 ratio for different angles. It is seen that the sensitivity b-parameter is highly dependent on the circumferential wave number as well. Change of the wave number yields a discontinuity in the sensitivity b-parameter, but as the angle increases the discontinuity becomes less pronounced. Furthermore, the wider the angle the lower the buckling load and the higher the sensitivity. The reason for that could be found in the plane surface of the shell, meaning with l/R_1 kept equal, as the angle increases so do the plane surface and the hydrostatic pressure.

The discontinuity in the sensitivity b-parameter is further investigated in Fig. 10 for a conical shell with $l/R_1 = 0.5$. It appears that

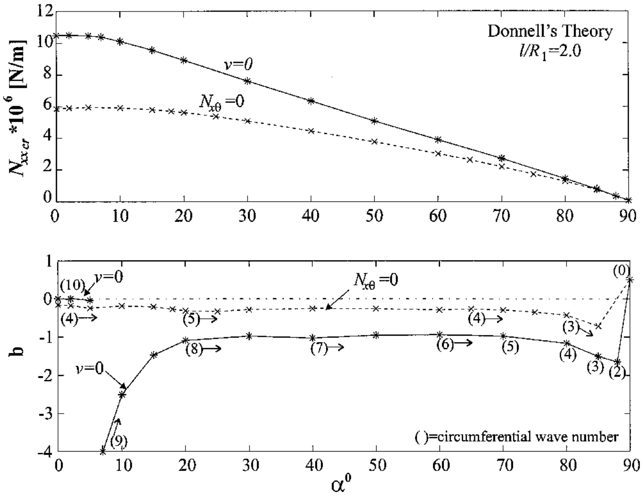

Fig. 7 Effect of in-plane boundary conditions on buckling load and on sensitivity b-parameter for simply supported conical shell under axial compression.

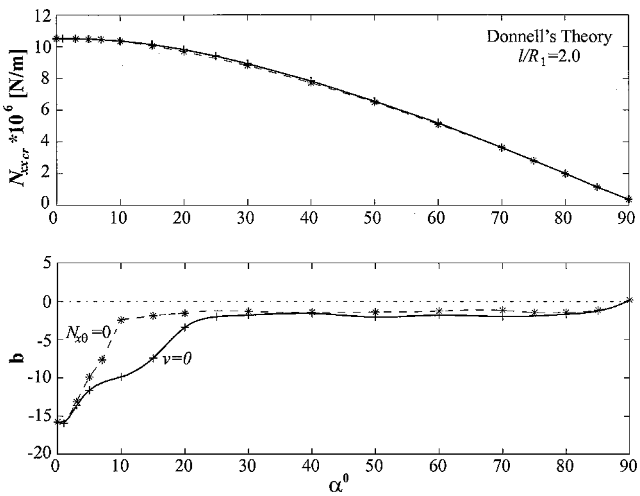

Fig. 8 Effect of in-plane boundary conditions on buckling load and on sensitivity b-parameter for clamped conical shell under axial compression.

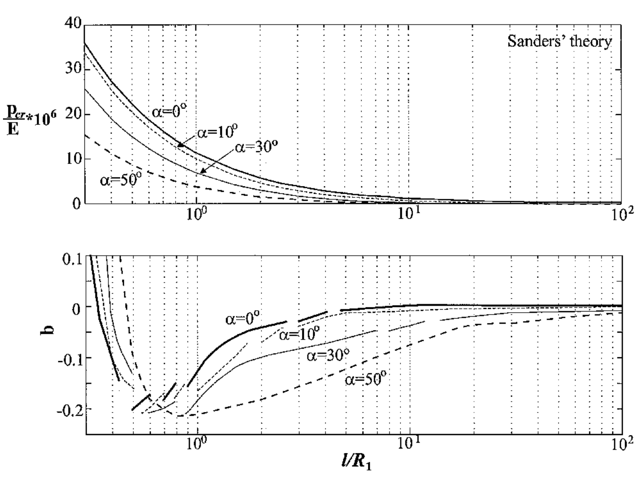

Fig. 9 Hydrostatic buckling load and sensitivity b-parameter vs l/R_1 ratio for simply supported ($u = N_{x\theta} = 0$) conical shell according to Sanders's theory.

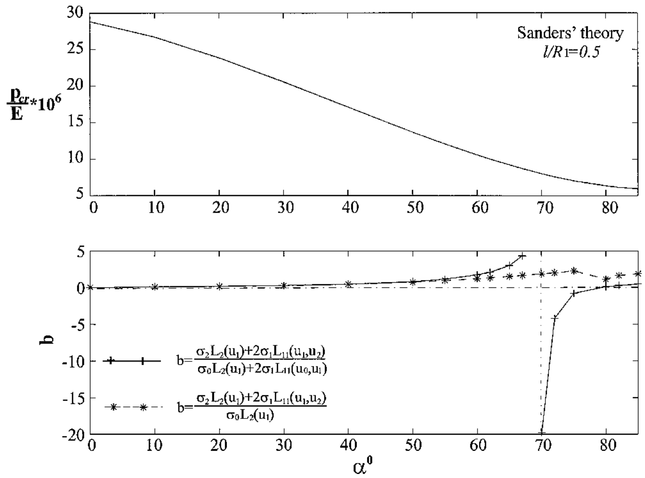

Fig. 10 Hydrostatic buckling load and sensitivity b-parameter vs cone vertex half-angle for simply supported ($N_{xx} = N_{x\theta} = 0$) conical shell according to Sanders's theory.

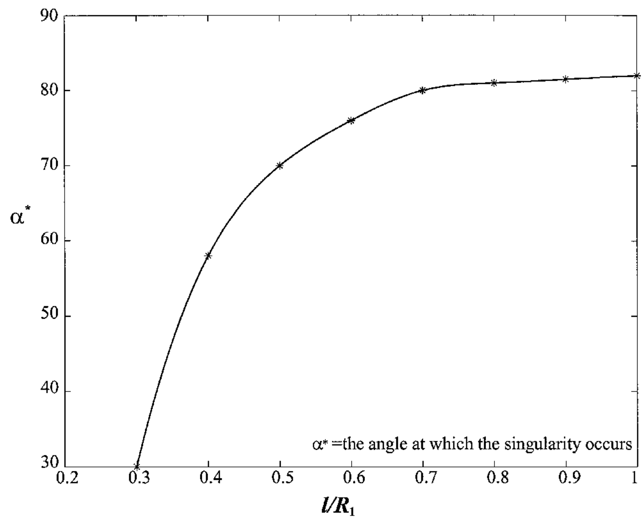

Fig. 11 Singularity location α vs l/R_1 ratio for conical shell under hydrostatic pressure.

there is a singularity at $\alpha = 70$ deg. At different l/R_1 the same singularity occurs at different angles (Fig. 11). In this case the singularity is due to the orthogonal variable in the denominator, $2\sigma_1 \cdot L_{11}(u_0, u_1)$, which in most cases is negligible but here is a major factor. Fitch[17] mentioned this phenomenon and proved its possibility. The solid and the dashed lines in Fig. 10 represent the sensitivity according to Eqs. (28) and (29), respectively. It seems that consideration of the sensitivity b-parameter under Budiansky's assumptions [Eq. (29)] is appropriate.

Conclusions

A procedure for the initial postbuckling behavior is presented for isotropic conical shells. The initial postbuckling behavior is examined in terms of the imperfection sensitivity of the shell via three theories: Donnell's, Sanders's, and Timoshenko's, which are the least, more, and most accurate, respectively.

1) The cone vertex half-angle α determines the characteristic behavior of the imperfection sensitivity, which is highly dependent on it. Generally, the angle scale could be divided into three zones. The most sensitive behavior is observed up to $\alpha = 20$ deg, beyond which the parameter decreases up to almost $\alpha = 80$ deg, down to complete insensitivity for an annular plate.

2) The sensitivity b-parameter is also highly dependent on the circumferential wave number. The behavior is characterized by transition from different circumferential wave number, which in some cases yields a discontinuity in the sensitivity b-parameter.

3) The sensitivity is also affected by the accuracy of the shell theory used in the analysis, namely the more accurate the theory, the lower the sensitivity. The most significant discrepancy occurs at small angles and in some cases only in a cylindrical shell.

4) The level of sensitivity depends on the shell parameters: the angle, the length-to-radius ratio (the longer the shell, the lower the sensitivity), and the boundary conditions. However, increase of the buckling load by optimizing one of these parameters may increase the sensitivity to some extent as well.

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